A Note on Gaussian Maxima via Effective Rank

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Abstract

In this note, we record a simple dimension-free upper bound for the maximum of a centered Gaussian random vector, stated in terms of the effective rank of its covariance matrix.

1 Main Result

Theorem 1.1. *Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a centered Gaussian random vector with covariance matrix Σ. Then*

$$\mathbb{E} \max_{1 \le i \le n} |X_i| \;\le\; C\sqrt{\|\Sigma\| \log(r(\Sigma)+1)}, \qquad r(\Sigma) := \frac{\mathrm{Tr}(\Sigma)}{\|\Sigma\|},$$

where $\|\Sigma\|$ denotes the operator norm of Σ and $C > 0$ is a universal constant.

We give two proofs of Theorem 1.1.

Proof of Theorem 1.1: Method 1. Let $g \sim \mathcal{N}(0, I_n)$ and write $X = \Sigma^{1/2} g$. Set $v_i := \Sigma^{1/2} e_i$ and

$$T := \{\pm v_1, \dots, \pm v_n\} \subset \mathbb{R}^n.$$

Then

$$\max_{1 \le i \le n} |X_i| = \max_{1 \le i \le n} |\langle g, v_i \rangle| = \sup_{t \in T} \langle g, t \rangle,$$

hence

$$\mathbb{E} \max_{1 \le i \le n} |X_i| = w(T) := \mathbb{E} \sup_{t \in T} \langle g, t \rangle,$$

the Gaussian width of T.

By Dudley's entropy integral bound, there exists a universal constant $C_0 > 0$ such that

$$w(T) \le C_0 \int_0^{\mathrm{diam}(T)} \sqrt{\log N(T, \varepsilon)}\, d\varepsilon, \tag{1.1}$$

where $N(T, \varepsilon)$ is the ε-covering number of T in Euclidean norm and $\mathrm{diam}(T) := \sup_{s,t \in T} \|s - t\|_2$. Since $\|v_i\|_2^2 = e_i^\top \Sigma e_i = \Sigma_{ii} \le \|\Sigma\|$, we have $\mathrm{diam}(T) \le 2\sqrt{\|\Sigma\|}$.

Fix $\varepsilon \in (0, 2\sqrt{\|\Sigma\|}\,]$ and define

$$S(\varepsilon) := \{i \in [n] : \; \|v_i\|_2 > \varepsilon\}.$$

Using $\sum_{i=1}^n \|v_i\|_2^2 = \mathrm{Tr}(\Sigma)$, we get

$$|S(\varepsilon)|\,\varepsilon^2 < \sum_{i\in S(\varepsilon)} \|v_i\|_2^2 \le \mathrm{Tr}(\Sigma), \qquad \text{hence} \qquad |S(\varepsilon)| \le \frac{\mathrm{Tr}(\Sigma)}{\varepsilon^2}.$$

Now cover T by Euclidean balls of radius ε: all points $\pm v_i$ with $\|v_i\|_2 \le \varepsilon$ are covered by the single ball $B(0,\varepsilon)$, while each $\pm v_i$ with $\|v_i\|_2 > \varepsilon$ is covered by its own ball. Therefore

$$N(T,\varepsilon) \le 1 + 2|S(\varepsilon)| \le 1 + \frac{2\,\mathrm{Tr}(\Sigma)}{\varepsilon^2}. \tag{1.2}$$

Plugging (1.2) into (1.1) and using $\mathrm{diam}(T) \le 2\sqrt{\|\Sigma\|}$ yields

$$\mathbb{E}\max_{1\le i\le n} |X_i| \le C_0 \int_0^{2\sqrt{\|\Sigma\|}} \sqrt{\log\Big(1+\frac{2\,\mathrm{Tr}(\Sigma)}{\varepsilon^2}\Big)}\, d\varepsilon.$$

Make the change of variables $\varepsilon = 2\sqrt{\|\Sigma\|}\,t$ for $t \in (0,1]$ to obtain

$$\mathbb{E}\max_{1\le i\le n} |X_i| \le C_1\sqrt{\|\Sigma\|} \int_0^1 \sqrt{\log\Big(1+\frac{c\,r(\Sigma)}{t^2}\Big)}\, dt,$$

for universal constants $C_1, c > 0$. Finally, splitting the integral at $t_0 := (1+r(\Sigma))^{-1/2}$ and using that $\log(1+c\,r/t^2) \lesssim \log(1+r) + \log(1/t)$, one checks that

$$\int_0^1 \sqrt{\log\Big(1+\frac{c\,r(\Sigma)}{t^2}\Big)}\, dt \;\lesssim\; \sqrt{\log(1+r(\Sigma))}.$$

Combining the above bounds completes the proof. □

Proof of Theorem 1.1: Method 2. Let $\Sigma = U\Lambda U^\top$ be the spectral decomposition of Σ, where $U \in \mathbb{R}^{n\times n}$ is orthogonal and

$$\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n), \qquad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0.$$

Let $Y = (Y_1,\dots,Y_n)$ be a mean-zero Gaussian random vector with covariance matrix Λ.

By [1, Lemma 1], we have the following comparison inequality

$$\mathbb{E}\max_{1\le i\le n} X_i^2 \;\le\; \mathbb{E}\max_{1\le i\le n} Y_i^2.$$

Moreover, by [3, Lemmas 2.3 and 2.4],

$$\mathbb{E}\max_{1\le i\le n} Y_i^2 \asymp \max_{1\le i\le n} \lambda_i \log(i+1).$$

We now relate the right-hand side to the effective rank $r(\Sigma)$. Since $\lambda_1 = \|\Sigma\|$ and $\mathrm{Tr}(\Sigma) = \sum_{i=1}^n \lambda_i = r(\Sigma)\,\lambda_1$, an estimate on nonincreasing sequences implies

$$\max_{1\le i\le n} \lambda_i \log(i+1) \le \lambda_1 \log\Big(\frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{\lambda_1} + 1\Big) = \|\Sigma\|\log(r(\Sigma)+1).$$

Combining the above inequalities and using Jensen's inequality, we obtain

$$\mathbb{E}\max_{1\le i\le n}|X_i| \le \sqrt{\mathbb{E}\max_{1\le i\le n} X_i^2} \lesssim \sqrt{\|\Sigma\|\log\big(r(\Sigma)+1\big)},$$

which completes the proof. □

Remark 1.2. Theorem 1.1 yields a convenient *one-sided* estimate for the maximum of a Gaussian random vector; in general, such a bound cannot be reversed. For matching upper and lower bounds for suprema of Gaussian processes, we refer to Talagrand's majorizing measure theorem [2]. □

References

[1] Stephane Mallat and Ofer Zeitouni. A conjecture concerning optimality of the Karhunen-Loeve basis in nonlinear reconstruction. *arXiv preprint arXiv:1109.0489*, 2011.

[2] Michel Talagrand. *Upper and lower bounds for stochastic processes: decomposition theorems*, volume 60. Springer Nature, 2022.

[3] Ramon Van Handel. On the spectral norm of gaussian random matrices. *Transactions of the American Mathematical Society*, 369(11):8161–8178, 2017.